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Dynamic scaling on fractals with sublattices

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Abstract. Dynamic scaling techniques are developed for antiferromagnets and multiatomic lattices. The methods are applied to antiferromagnets and triatomic lattices on Sierpinski gasket fractals, and to ferro- and antiferromagnets on Berker hierarchical lattices. Relationships between the frequencies of ferro- and antiferromagnets on the same fractal and of mono- and multiatomic fractal systems are given. In particular the dynamic exponents z_A and z_F of antiferromagnetic and ferromagnetic Sierpinski gaskets or Berker lattice fractals are shown to satisfy $z_A = \frac{1}{2} z_F$.

1. Introduction

In this paper length scaling techniques [1-9] are developed for the following dynamical processes on fractals [10]: ferro- and antiferromagnetic spin dynamics and monatomic and multiatomic lattice vibrations. Apart from the generalisation of existing methods to multi-sublattice systems, and the first treatment of ferromagnetic and antiferromagnetic spin waves on the Berker hierarchical lattice, of antiferromagnetic spin dynamics on a Sierpinski gasket fractal, and of the lattice vibrations of a triatomic Sierpinski gasket, the paper also includes an investigation of interrelationships between the dynamical behaviour of single and multiple sublattice fractal systems. For all the fractals studied, which include non-uniform systems with two and three sublattices, the dynamic critical exponents z_F and z_A of ferro- and antiferromagnetic spin waves are found to satisfy the same relationship

$$z_A = \tfrac{1}{2} z_F \tag{1}$$

which holds in simple regular lattices; this is so, even though (unlike the simple case) these exponents are non-trivial (not integers or rational fractions) and describe anomalous dynamics arising from the self-similarity of the fractals [4, 7, 9]. A relationship is also found between the exponents describing the lattice vibrations of a multiatomic and a monatomic fractal.

The present interest in non-random fractals [10] arises principally because, despite being constructed in a simple (recursive) manner, they display properties of scale invariance and non-uniformity. These are respectively the essential properties of structures at continuous phase transitions and of amorphous or random systems. An example where both properties are used is in representing by fractals [11] (e.g. the Sierpinski gasket) the infinite cluster or its backbone at the percolation threshold.

Among dynamical properties of fractals so far investigated are: ferromagnetic spin waves [1, 7], monatomic lattice vibrations [4, 8], isotropic [3, 4, 9], anisotropic and

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biased diffusion [12] on Sierpinski gasket fractals; and monatomic [8] and mixed [13] lattice vibrations on Berker lattices. The present paper achieves some extension of this list and at the same time provides a generalisation of some of the existing methods and an investigation of interrelationships between dynamic exponents (beyond that which occurs, as in some cases listed above, because of an exact equivalence between the equations of motion of the systems [1]).

The usual wavevector representation methods of dealing with dynamics and other properties of regular lattices (in which relationships like (1) arise in a very obvious way) cannot be applied to fractals because of their non-uniform character. However, fractals are amenable to length scaling treatments in position space because these methods involve a 'decimation' process which is the inverse of that used in the construction of the fractals, namely recursive insertion of structure on a smaller scale [10].

The scaling procedures used here are extensions of those introduced previously [1]. A number of different extensions are possible, each with advantages in certain specific situations. To introduce these developments and distinguish between them a preliminary discussion is given for chain systems. The main part of the paper then follows, in which Sierpinski and Berker hierarchical lattices are discussed in turn.

2. Dynamic scaling methods for antiferromagnetic and diatomic chains

The idea of the length scaling method is to eliminate from the equations of motion some fraction of the sites and hence achieve a dilatation by scale factor b. The associated transformation of parameters is the renormalisation group recursion relation. For simple cases like ferromagnets, diffusion or monatomic lattice vibrations it is usual to eliminate every other site ($b=2$). But for antiferromagnets or diatomic lattices that convert the system to a ferromagnet or monatomic lattice, i.e. to a different system, the subsequent scalings (ferromagnet to ferromagnet, etc) give the recursion relation. This method is hereafter denoted by I.

For example, the application of method I to an antiferromagnetic chain with spins S_l at sites l starts from the equation of motion

$$[-2+(-1)^l\omega]S_l^+ = S_{l+1}^+ + S_{l-1}^+ \tag{2}$$

where ω is strictly the frequency divided by the exchange constant. The main stages are as follows: elimination of sites of one sublattice (l odd, say) gives a ferromagnet with spins at sites $i=2l$ of the original system, with equation of motion

$$(2-\omega_F)S_i^+ = S_{i+1}^+ + S_{i-1}^+ \tag{3}$$

involving the effective frequency $\omega_F=\omega^2$ (which is already enough to imply $z_F=2z_A$). Subsequent scaling by $b=2$ gives the usual result [1, 7] for the transformed (ferromagnetic) frequency

$$\omega_F' = 4\omega_F - \omega_F^2. \tag{4}$$

Linearisation of (4) about the zero-frequency fixed point leads, in the usual way [1], to $\omega_F \propto k^{z_F}$ at small values of the reduced wavevector k, where $z_F=2$. The quadratic relationship between ω and ω_F then gives $\omega \propto k^{z_A}$ with

$$z_A = \tfrac{1}{2}z_F = 1. \tag{5}$$

The complete dispersion relationship for the ferromagnet can be obtained directly from (4) [1], from which it follows that for the antiferromagnet

$$\omega^2 = 2(1-\cos k). \tag{6}$$

An alternative method (method II) is to dilate by an odd scale factor b, in which case the antiferromagnet (or diatomic lattice) goes into a scaled system of similar type. This is preferable to method I, which loses all information about the relationships between sublattices, and is necessary for fractals whose construction involves an odd scale factor. Since spins on both sublattices remain, the relationships of phases and also amplitudes of spin deviations on the two sublattices occur. As usual the elimination preserves the absolute phase relationship between the retained spins. However, the occurrence of the amplitude ratio r on the two sublattices is a new feature which can be handled in either of the following ways (methods II(a), II(b)): (a) allow the amplitude ratio to change under scaling, or (b) introduce a new scaling parameter so that the amplitude ratio can be preserved under scaling.

For the antiferromagnetic chain, application of method II(a) gives the following pair of scaling equations for characteristic frequency ω and amplitude ratio r, for dilatation by $b=3$:

$$\omega' = 3\omega - \omega^3 \tag{7}$$

$$\left(\frac{2-\omega'}{2+\omega'}\right) r'^2 = \left(\frac{2-\omega}{2+\omega}\right) r^2. \tag{8}$$

Linearisation of (7) yields directly $z_A = 1$. Alternatively, (7) and (8) can be solved exactly to yield the usual results [14]

$$\omega = 2\sin k \qquad r^2 = A(1+\sin k)/(1-\sin k)$$

where k is the reduced wavevector and A is a constant.

In method II(b), equation (2) needs generalisation since its original form is not maintained under scaling. A convenient form to use replaces the bracket on the left-hand side by the different factors $(-2-\alpha)$ (l odd) or $(-2+\beta)$ (l even) for the two sublattices, extending the parameter space from one parameter (ω) to two (α, β). The resulting scaling equations for scale factor $b=3$ are

$$\alpha' = 5\alpha - 4\beta + 2\alpha^2 - 4\alpha\beta - \alpha^2\beta \tag{9}$$

$$\beta' = 5\beta - 4\alpha - 2\beta^2 + 4\alpha\beta - \beta^2\alpha. \tag{10}$$

The linearised form of these equations has eigenvalues $\lambda = 9, 1$. The first of these eigenvalues is associated with the scaling of the combination $\gamma \equiv 2(\alpha-\beta) - \alpha\beta$ (the exact scaling equation $X' = X(X-3)^2$, where $X \equiv (2+\alpha)(2-\beta) \equiv 4+\gamma$ follows directly from (9) and (10) and it has fixed point $X^* = 4$ and eigenvalue $\lambda_X = 9$). The second eigenvalue $\lambda = 1$ results because $(2+\alpha)/(2-\beta)$ is exactly invariant under scaling. Because of the symmetry $\alpha = \beta$ present in the original system of equations, γ is initially of order ω^2, so the dynamic exponent z_A is related to λ_X by $b^{z_A} = \sqrt{\lambda_X} = 3$, leading again to $z_A = \frac{1}{2} z_F = 1$. In this method the result is seen to arise from a fortuitous vanishing of the linear term in the scaling field γ, brought about by the symmetry of the initial conditions. Whether this is a universal feature will later be tested with a calculation on the hexagonal Berker lattice. Clearly, method II(b) is closely related to II(a); however, it will be more convenient for use in obtaining the scaling for the Berker lattice.

Before treating the fractals we first briefly indicate how the above procedures can be used to scale diatomic vibrating chains. In this case the original equation of motion is similar to (2) but with the bracket on the left-hand side replaced by $(2-m\omega^2)$ or $(2-M\omega^2)$ depending on whether l is a site with reduced mass m or M respectively. So, method I is simple to apply, converting the system to an equivalent monatomic chain with a mass × (frequency)^2 parameter equal to $2\omega^2(m+M)-mM\omega^4$, which thereafter scales in the same way as ω_F in (4). Method II(a) also applies in a straightforward manner leading to equations identical to (7) and (8) but with $[4-(2-m\omega^2)(2-M\omega^2)]^{1/2}$ and $r^2(2-m\omega^2)/(2-M\omega^2)$ replacing ω and $r^2(2-\omega)/(2+\omega)$ respectively. Equivalent results are also obviously obtained via method II(b), by making the substitutions $\alpha=-m\omega^2$, $\beta=M\omega^2$.

3. Dynamics of three-sublattice Sierpinski gasket fractal

The Sierpinski gasket fractal is now considered. For this system the dynamics of ferromagnetic spin waves [1, 7] and the closely related diffusion problem [3, 4, 9] have been earlier treated by the scaling method. We now discuss the antiferromagnetic case, taking to be specific a system of classical Heisenberg spins on the triangular Sierpinski gasket. This system can be divided into three sublattices, A, B, C, as indicated in figure 1(a). It is easy to check that with antiferromagnetic nearest-neighbour coupling, in the lowest energy state the spins on sublattice α lie along the unit vector $\boldsymbol{e}_\alpha$ where $\boldsymbol{e}_A+\boldsymbol{e}_B+\boldsymbol{e}_C=0$ (i.e. neighbouring spins are at an angle of $2\pi/3$ in the ground state); this can also easily be seen to be a stable configuration, though that will anyway be confirmed by the real nature of the eigenfrequencies. For a small deviation from its ground state direction $\boldsymbol{e}_\alpha$ a particular (edge) spin $\boldsymbol{S}_\alpha$ on sublattice α can be represented by $\boldsymbol{S}_\alpha=\boldsymbol{e}_\alpha+\boldsymbol{e}_\alpha\times\boldsymbol{u}_\alpha$ where $\boldsymbol{u}_\alpha\cdot\boldsymbol{e}_\alpha=0$ and $\boldsymbol{u}_\alpha$ is small. Then, to first order in $\boldsymbol{u}$ the equations of motion become

$$i\omega\boldsymbol{e}_\alpha\times\boldsymbol{u}_\alpha\equiv\boldsymbol{e}_\alpha\times\dot{\boldsymbol{u}}_\alpha=\tfrac{1}{2}\sum_{\nu=1,2}(2\boldsymbol{u}_\alpha-\boldsymbol{u}_{\beta\nu}-\boldsymbol{u}_{\gamma\nu}) \tag{11}$$

where α, β, γ are the three different sublattices, and $\nu=1,2$ label the two neighbours on a given sublattice (as indicated in figure 1(b)). Or, putting $\boldsymbol{u}_\alpha=X_\alpha\hat{\boldsymbol{x}}+Y_\alpha\hat{\boldsymbol{x}}\times\boldsymbol{e}_\alpha$ where $\hat{\boldsymbol{x}}$ is a unit vector perpendicular to the plane containing $\boldsymbol{e}_A$, $\boldsymbol{e}_B$, $\boldsymbol{e}_C$ the equations can be reduced to

$$(4-\Omega)Z_\alpha=\sum_{\nu=1,2}(Z_{\beta\nu}+Z_{\gamma\nu}) \tag{12}$$

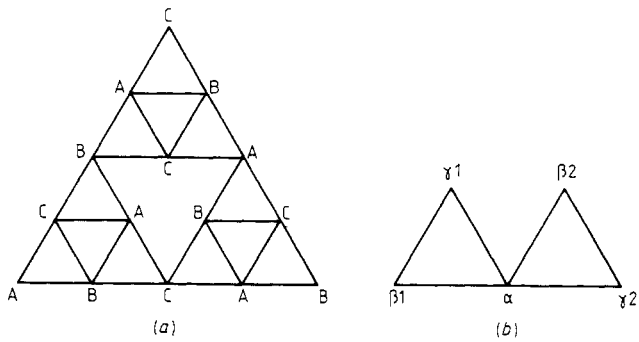

Figure 1. (a) Labelling of sublattices for a Sierpinski gasket; (b) labelling of the nearest neighbours of site α.

where

$$Z \equiv X + i\Omega Y/4\omega \qquad \Omega \equiv 2(3-\sqrt{9-2\omega^2}). \tag{13}$$

The equation of motion now has a similar form to that for the ferromagnetic Sierpinski gasket at effective frequency Ω. The usual decimation procedure for that case [1, 7] can therefore be applied to achieve a dilatation by $b=2$. The resulting scaling of Ω is

$$\Omega' = 5\Omega - \Omega^2. \tag{14}$$

Linearising (14) about the fixed point $\Omega^* = 0$ yields the dynamic exponent for the ferromagnetic Sierpinski gasket

$$z_F = \ln 5/\ln 2. \tag{15}$$

From (13), the region near $\Omega^* = 0$ corresponds to ω small where $\omega \propto \Omega^{1/2}$. Thus, because of the mapping to the equivalent ferromagnetic system, the dynamic exponent for the antiferromagnetic Sierpinski gasket is half of that for the ferromagnetic case:

$$z_A = \tfrac{1}{2} z_F = \tfrac{1}{2}\ln 5/\ln 2. \tag{16}$$

We now outline procedures that can be used to discuss the lattice dynamics of Sierpinski gaskets with three different masses m_A, m_B, m_C respectively, on sublattices A, B, C of figure 1(a). For simple spring forces (no bond-bending forces) the original equations of motion are of the form (12), but with Ω replaced by $m_\alpha \omega^2$ and Z now denoting the atomic displacement. For $m_\alpha = m$ (monatomic case) we recover a scaling like (14). But for different sublattice masses, a much more complicated set of scaling equations applies, rather similar in structure to those describing anisotropic diffusion on the fractal [12]. Linearisation about the low frequency fixed point yields eigenvalues 5, $1, \frac{1}{5}, -\frac{23}{5}$. The eigenvector for the first eigenvalue (which is that describing the monatomic case) always has overlap with physically attainable vectors (while that for the last eigenvalue never does). So the low frequency dynamics of the multiatomic fractal has the same exponent as that for the monatomic one (which, in turn, is the same as that given in (16) for the antiferromagnetic case).

4. Ferro- and antiferromagnetic dynamics on Berker lattice fractals

We now consider ferromagnetic and antiferromagnetic dynamics on the Berker lattice fractal. The usual Berker lattice was introduced as the system for which the Migdal-Kadanoff recursion relations for the two-dimensional Ising model are exact [15]. It is generated recursively by replacing each bond of the $(n-1)$th generation by a rhombus to form the nth generation. In this form the Berker lattice is not a sensible test of the relation (1) because there is an asymmetry between the up and down sublattices. Instead we introduce a variation which provides a fair test of the relation: in the construction a hexagon is used rather than the usual rhombus. Since it is now impossible to map the antiferromagnet directly onto the ferromagnet (as was the case for the chain, etc) there is perhaps the possibility of violating (1).

We first treat the ferromagnet. If 2^{n_i+1} denotes the number of bonds incident on the site i, n_i depends on the generation of the lattice, increasing by one in going from the $(n-1)$th to the nth generation. The equations of motion can be written in the following form

$$(2^{n_i+1}u - \omega)S_i^+ = \sum_{j=1}^{2^{n_i+1}} S_j^+. \tag{17}$$

The sum is over the 2^{n_i+1} sites which are connected to i by a direct bond. The parameters u and ω turn out to be both sufficient and necessary to perform an exact scaling if the initial conditions are taken to be $u=1$ (because of the Goldstone symmetry in the problem) with ω arbitrary; thus an extension of parameter space like that in method II(b) is required. The sites whch are removed at each stage only have two nearest neighbours and hence are described by the equations of motion

$$(2u-\omega)S_i^+ = S_{j_1}^+ + S_{j_2}^+. \tag{18}$$

Eliminating the lowest level sites in the usual way leads to the following scaling relations for u and ω:

$$u' = u[(2u-\omega)^2 - 1] - (2u-\omega) \tag{19}$$

$$\omega' = \tfrac{1}{2}\omega[(2u-\omega)^2 - 1]. \tag{20}$$

The eigenvalues of these equations linearised at the fixed point $(\omega^*, u^*) = (0, 1)$ are $\lambda = \frac{3}{2}$, 9. The larger of these determines the low frequency scaling of ω, there being no special symmetries in the problem to cause the vanishing of the term linear in ω in the associated scaling field. Hence the dynamic exponent is

$$\omega_F = \ln 9/\ln b. \tag{21}$$

The factor b is somewhat ambiguous for the Berker lattice, since there is no natural embedding of it in Euclidean space, and so we leave the result in the form (21) for later comparison with the antiferromagnetic result.

The exact scaling of the antiferromagnet on the same hexagonal Berker lattice is found by following the evolution of a set of equations with three independent scaling variables u, v, ω (a generalisation of method II(b)):

$$(-2^{n_i+1}v - \omega)S_i^+ = \sum S_j^+ \tag{22}$$

when i is on the up sublattice and

$$(-2^{n_j+1}u + \omega)S_j^+ = \sum S_i^+ \tag{23}$$

when j is on the down sublattice.

The initial conditions for the scaling are $u = v = 1$ with ω arbitrary. As before, the summation on the right of the equations is over the 2^{n+1} nearest-neighbour spins. Elimination of the lowest level sites gives the recursion relations for the renormalisaed parameters:

$$\omega' = \tfrac{1}{2}\omega[(2u-\omega)(2v+\omega) - 1] \tag{24}$$

$$u' = u[(2u-\omega)(2v+\omega) - 1] - (2u-\omega) \tag{25}$$

$$v' = v[(2u-\omega)(2v+\omega) - 1] - (2v+\omega). \tag{26}$$

These have fixed point $(\omega^*, u^*, v^*) = (0, 1, 1)$ and eigenvalues 9, $\frac{3}{2}$, 1. The associated linear scaling fields (to first order in the variables ω, $\eta \equiv u - u^*$, $\psi \equiv v - v^*$) are respectively $\eta + \psi$, ω, $\eta - \psi - 4\omega$. Since for the antiferromagnet one is interested in scaling from the initial conditions $\eta = \psi = 0$ there is a problem completely analogous to that for the antiferromagnetic chain, so non-linear scaling fields are needed to extract the exponent z. A short calculation gives second-order approximations to the scaling fields and shows that ω^2 initially has a non-zero projection on the non-linear scaling field $\psi + \eta + \frac{1}{4}\omega\psi - \frac{11}{27}\omega^2 - \psi^2$ associated with the eigenvalue $\lambda = 9$. It follows that the antiferromagnetic dynamic exponent z_A is given by

$$2z_A = \ln 9/\ln b. \tag{27}$$

Hence, using (21), this Berker lattice, like the Sierpinski gasket fractal, satisfies the relationship (1).

This result extends to more general 'hexagonal' Berker lattices: when the number of parallel three-bond branches used at each stage of the construction of the hierarchical lattice is an arbitrary integer n, the discussion given above for the special case $n=2$ easily generalises to provide exactly similar scaling equations to (19), (20) and (24)-(26) except for the replacement of the factor $\frac{1}{2}\omega$ by ω/n in (20) and (24). (For $n=1$, the generalised equations reduce correctly to those, (9) and (10), obtained earlier by applying method II(b) to the antiferromagnetic chain.) The eigenvalues are then $\lambda=9$, $3/n$ (and, in the case of the antiferromagnet, $\lambda=1$). Thus for any $n\geqslant 1$ the eigenvalue $\lambda=9$ again determines the dynamic exponents z_F, z_A, the latter because the associated scaling field is essentially ω^2. Hence again $z_A=\frac{1}{2}z_F$. (An interesting special case is $n=3$: here ω and $\eta-\psi$ are two independent marginal fields in the case of the antiferromagnetic treatment.)

5. Discussion

In conclusion, scaling methods have here been developed for sublattice systems, and have been applied to various non-random fractals. In all the examples considered, simple relationships between dynamic exponents of ferro- and antiferromagnets, and between those of monatomic and multiatomic lattices have occurred. In some cases the relationship between dynamic behaviour extends to all frequencies. Though it is tempting to conjecture that such relationships hold for systems of other types than the fractals here considered we will argue elsewhere [16] that the relation (1) is violated for random fractal systems, such as the percolation cluster at p_c, because of a breakdown of the local symmetry between up and down sublattices.

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